Quiz 1 Solution

October 12, 2018

Problem. Let V_i for $i = 1,\ 2,\ 3,$ and 4 denote the vertices of a unit regular tetrahedron. Compute the number of walks along four edges of the tetrahedron of length 4 that begin at V_i and end at V_j.

Solution. As each vertex of a tetrahedron T is connected to any other vertex by an edge of T, the adjacency matrix A of the graph formed from the edges of T is given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

By Proposition 1.2.23 it follows that the number of walks along four edges of the tetrahedron of length 4 that begin at V_i and end at V_j is given by the ij^{th} entry of A^4. But

$$A^2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix},$$

and, as $A^4 = (A^2)^2$,

$$A^4 = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 20 & 20 & 20 \\ 20 & 21 & 20 & 20 \\ 20 & 20 & 21 & 20 \\ 20 & 20 & 20 & 21 \end{bmatrix}.$$

Therefore, if $i = j$ there are 21 walks along four edges of the tetrahedron of length 4 that begin at V_i and end at V_j, and if $i \neq j$, there are 20 such walks.